Decomposition of $C_r[\overline{K}_m]$ into sunlet graphs

A. D. Akinola

Abstract— In this paper,it has been proved that $C_r[\overline{K}_m]$ (which is the wreath product of cycle and complement of a complete graph K_n can be decompose into sunlet graph L_n, n = rm if and only if $n \mid rm^2$ and m is even.

2000 Mathematics subject classification: 05B30, 05C70

Keywords: Graph decomposition,sunlet graph, wreath product of cycle and complement of a complete graph.

1 INTRODUCTION

We begin with a few definitions.A graph $C_r[\overline{K}_m]$ is a graph which arises from the cycle C_r by replacing each vertex *x* by *m* independent vertices and each edge *xy* by $x'y', x'y'',..., x'y^{m1}; x''y',..., x''y^m ,..., x^m y',..., x^m y^m$.

A graph *G* with *q* edges is said to be decomposable into graph *H* if it can be written as the union of edge-disjoint copies of *H* so that every edge in *G* belongs to one and only one copy of *H* . Although much work has gone into decomposition of complete graphs into *k* cycles(See [4] for good survey).Alspach and Gavlas [1] have shown the necessary and sufficient condition for C_m to decompose the complete graph K_n when *m* and *n* are either both odd and even.M. Sajna [2] have shown that for the case when *m* and *n* are of opposite parity.N.J. Cavenagh and etal [3] have worked on decomposition of complete multipartite graphs into cycles of even length.D. Froncek etal worked on decomposition of complete graph into blownup cycles $C_{r[}[\overline{K}_2]$. R. Haggkvist [5] gave lemma on cycle decomposition. N.J. Cavenagh [10] have shown that the graph $C_r[\overline{K}_m]$ can be decompose into cycle C_r . R. Laskar [11] has proved that the graph $C_r[\overline{K}_m]$ can be decompose into Cycles C_{rm}. R.Anitha etal [7] worked on N-sun (sunlet graph) decomposition of complete ,complete Bipartite and some harary graphs . Little attention has been paid to the same problem for the graph $C_r[\overline{K}_m]$. A significant and useful result is that of Sotteau [8] who discover necessary and sufficient conditions for the decomposition of complete bipartite graphs into *k* cycles.

2 DEFINITIONS AND TERMINOLOGY

2.1 Definition

Sunlet graph L_r is a graph with cycle $C_{r/2}$ whereby each vertex of the cycle is attached to one pendant vertex .Each sunlet graph contains *r* vertices with *r* edges.

2.2 Definition

We will define a function on the vertices of a graph *G* = $C_r[\overline{K}_m]$ as $c : V(G) \to x_{N\times N}$ i.e for every vertex $x \in G$, $c(x) = x_{ij}$ where $i, j \in N$.

2.3 Definition

All graphs in this paper are simple .We use K_m to denote the complete graph on *m* vertices. $\overline{K}_m$ denote the complement of K_m. An *r*-cycle is a cycle

ISSN 2229-5518

of length *r* and is denoted by C_r. Let $C_r[\overline{K}_m]$ stand for a cycle in which every vertex is replaced by *m* isolated vertices and every edge by $K_{m,m}$ and $C_r[\overline{K}_m]$ contains vertex x_{ij} ; *i* = 1,…, *r* and *j* = 0,…,*m*-1. Let L_n be a sunlet graph with *n* vertices and *n* edges.

2.4 Definition

Let *G* be a graph without loops and $L = (H_1, H_2, ..., H_m)$ a list of graphs.The list *L* is proper if G has a subgraph isomorphic with H_i, *i* = 1, 2,…,*m* and $|E(G)| = \sum_{i=1}^{m} |E(H_i)|$ where *E*(*G*) denotes the edge set of *G* .The list *L* is said to pack *G* if *G* is the edge-disjoint union of the graphs $G_1, G_2, ..., G_m (G = G_1 \oplus G_2 \oplus ... \oplus G_m)$ where G_i is isomorphic with H_i for *i* = 1,2,…,*m* .An even list where each entry occurs an even number of times. The graph *G* is said to have an *M*-decomposition if the proper list (*M,M,…,M*) packs *G*,that is *M*|*G*.

3. DECOMPOSITION OF $C_r[\overline{K}_m]$ INTO EDGEDISJOINT SUNLET GRAPHS

3.1 Lemma

The graph $C_r[\overline{K}_2]$ can be decompose into 2 sunlet graphs with 2*r* vertices.

Proof.

From the definition of the graph $C_r[\overline{K}_2]$ (denoted by $C_r[2]$), each vertex *x* in C_r is replaced by a pair of two independent vertices x', x'' and each edge *xy* is replaced by four edges $x'y', x'y'', x''y', x''y''$ First form 2 cycles C'_r and C''_r with vertex set x'_i and x''_i] respectively, *i* = 1,…, *r*. Define a mapping *Á* by $\phi(x'_i) = x''_{i+1}$ and $\phi(x''_i) = x'_{i+1}$, *i* is calculated modulo *r*. $\phi(x'_i)$ and $\phi(x''_i)$ represent pendant vertices.Attached each pendant vertices $\phi(x'_i)$ to each vertex x'_i in C'_r and $\phi(x''_i)$ to each vertex x''_i in C''_r , C''_r with pendant vertices $\phi(x'_i), \phi(x''_i)$ attached respectively gives 2 sunlet graphs with 2*r* vertices.Hence $C_r[\overline{K}_2]$ can be decompose into 2 sunlet graphs with 2*r* vertices

.

3.2 Lemma

Let r,*m* be positive even integers such that *n* = *rm* then $C_r[\overline{K}_m]$ can be decompose into *m* sunlet graph L_r .

Proof:

First observe that $C_r[\overline{K}_2]$ can be decompose into 2 sunlet graphs with 2*r* vertices.Then set *m* = 2*t* and decompose $C_r[\overline{K}_t]$ into t^2 cycles C_r .Denote vertices in *i - th* part of $C_r[\overline{K}_t]$ by x_{ij} for *j* = 0, ,…,*t*- 1 and create *t* cycles $x_{ij}x_{2j}x_{3j}...x_{r-1j}x_{rj}$ for *j* = 0,1,…, *t*-1. Next combine these cycles into one cycle C_{rt} by replacing each edge $x_{1j}x_{2j}$ with $x_{1j}x_{2j+1}$ .Now apply mappings ϕ_s for *s* = 0, 1,…, *t*- 1 defined as follows:

$\phi_s(x_{ij}) = x_{ij}$ for *i* odd and

$\phi_s(x_{ij}) = x_{ij+s}$ for *i* even. This is the desired decomposition into cycles C_{rt} .Now take each cycle C_{rt} ,make it back into $C_{rt}[\overline{K}_2]$ and decompose $C_{rt}[\overline{K}_2]$ into 2 sunlet graphs $L_{2rt} = L_{rm} = L_n$ .Hence L_n decomposes $C_r[\overline{K}_m]$ for *r*,*m* even and *n* = *rm*.

Each graph $C_{rt}[\overline{K}_2]$ gives 2 sunlet graphs L_n and we have *t* graph $C_{rt}[\overline{K}_2]$ which implies that we have 2*t* = *m* sunlet graphs L_n .Hence $C_r[\overline{K}_m]$ decomposes into *m* sunlet graphs L_n .

3.3 Lemma

ISSN 2229-5518

If *r* is a positive odd integers and *m* a positive even integers such that *n* = *rm*,then sunlet graph L_n .decomposes $C_r[\overline{K}_m]$.

Proof:

We have *r*- 1 even since *r* is odd.Set *m* = 2*t*. First create *k* cycles C_{r-1} in $C_{r-1}[\overline{K}_m]$ as in lemma (2).Then take complete tripartite graph $K_{t,t,t}$ with partite sets $X_i = \{x_{ij}\}$ for *i* = 1, *r* – 1, *r* and *j* = 0, 1,…, *t*- 1 and decompose it into triangles using well known construction via latin square.i.e Construct $t \times t$ latin square and consider each element in the form (*a* ,*b*, *c*) where *a* denotes the row,*b* denotes the column and *c* denotes the entry with $0 \le a,b,c \le t-1$.Each cycle is of the form (1, *a*) , (*r*- 1, *b*) , (*r*, *c*)

Then for every triangle $x_{1a}x_{r-1b}x_{rc}$ replace the edge $x_{1a}x_{r-1b}$ in an appropriate C_{r-1} by the edges $x_{r-1b}x_{rc}$ and $x_{rc}x_{1a}$ to obtain cycle C_r :

Therefore we have $C_r \mid C_r[\overline{K}_t]$.

Case 1 :Suppose $\frac{m}{2}$ is even.

Join 2 cycles C_r into one cycle to give C_{2r} : i.e

$x_{ij}x_{2j}x_{3j}...x_{r-1j}x_{rj}x_{ij+1}x_{2j+1}...x_{rj+2}$,

$j = 0,1,...,t-2$

Now apply mapping ϕ_s for *s* = 0, 1,…, *t*- 1 defined as follows:

$\phi_s(x_{ij}) = x_{ij}$ for *i* odd and $\phi_s(x_{ij}) = x_{ij+s}$ for *i* even and *i* = *r*.

This is the desired decomposition into cycles C_{2r} .Now take each C_{2r} , make it into $C_{2r}[\overline{K}_2]$. and decompose it into two sunlet graphs L_{4r} . .

Case 2: Suppose $\frac{m}{2}$, is odd

Join the cycles *Cr* by replacing each edge $x_{1j}x_{2j}$ with $x_{1j}x_{2j+1}$. in order to create *t* cycles C_{rt} .Then apply mapping ϕ_s for *s* = 0, 1,…*t* - 1 defined as follows:

$\phi_s(x_{ij}) = x_{ij}$ for *i* odd and

$\phi_s(x_{ij}) = x_{i+j+s}$ for *i* even and *i* = *r*.m

Now take each C_{rt} ,make it into $C_{rt}[\overline{K}_2]$ and decompose it into two sunlet Graphs L_{2rt} $= L_{rm} = L_n$ Hence sunlet graph L_n decompose $C_r[\overline{K}_m]$ for *r* odd. Therefore for $m/2, r$ odd, $C_r[\overline{K}_m]$ can be decomposes into *m* sunlet graph L_n.

3.4 Lemma

If sunlet graph L_n decomposes $C_r[\overline{K}_m]$, *n* = *rm,* then *m* is even.

Proof:

Each vertex in $C_r[\overline{K}_m]$ has degree 2*m.* For L_n to decompose $C_r[\overline{K}_m]$ with *n* = *rm*,3*w*+*p* must be equal to 2*m* where w, *p* represent the number of times the vertex x_{ij} in $C_r[\overline{K}_m]$ appears in the decomposition with degree 3 and 1 respectively and *w* + *p* = *m* .Suppose *m* is odd,let there exist a vertex x_{ij} in $C_r[\overline{K}_m]$ such that it exist in each sunlet graph L_n that decomposes $C_r[\overline{K}_m]$.The sum of its degree in all the sunlet graphs L_n is 3*w*+*p*. In all the cases for the values of *w* and *p*, 3*w*+*p* 6= 2*m*.which implies that L_n does not decomposes $C_r[\overline{K}_m]$ for *m* odd but for all *m* even,3*w* + *p* = 2*m*.The result also follows from [8] for *r* > 2.Hence the result.

3.5 Theorem

If *G* = $C_r[\overline{K}_m]$ can be decompose into cycle C_r then *G*(2) can be decompose into $2m^2$ sunlet graphs with 2*r* vertices.

Proof:

From the previous observation all we need to show is that $C_r[2]$ decomposes into 2 copies of sunlet graphs with 2*r* vertices.By Lemma (3.1) , $C_r[\overline{K}_2]$ can be decompose into 2 sunlet graphs with 2*r* vertices.Hence we have $2m^2$ sunlet

ISSN 2229-5518

graph L_{2r} since we have m^2 cycles C_r from the cycle decomposition of $C_r[\overline{K}_m]$

3.6 Lemma

Necessary conditions for L_n to decompose $C_r[\overline{K}_m]$ whenever *n* = *rm* are:

(a) *m* must be positive even integers.

(b) $r \geq 3$

(c) The number of edges *n* divides the number of edges in $C_r[\overline{K}_m]$.

Proof:

(a) The result follows from lemma (3.4) that *m* must be positive integers.

(b) It follows from lemma (3.2) and (3.3).

(c) It follows from the fact that the number of edges *n* of L_n has to be a multiple of *m* since the total number of edges in $C_r[\overline{K}_m]$ is rm^2.Thus for such decomposition to exist, we have that *n*|*rm.*

3.7 Theorem

Let *m, n* be positive even integers and $r \geq 3$, *n* = *rm,*if L_n decomposes $C_r[\overline{K}_m]$ then the 2 sunlet graphs such that each vertex appears once with degree 3 and 1 in each sunlet graph L_n respectively is a 4 regular graph which can be decompose into 2 hamiltonian cycles.

Proof:

Suppose the sunlet graph L_n decomposes $C_r[\overline{K}_m]$,let *m* = 2*t*.Then by lemma (3.1) and lemma (3.2) and [8]*; Crt* | $C_r[\overline{K}_t]$.Let *G* = $C_r[\overline{K}_t]$. then by theorem (3.4) *G*(2) can be decompose into sunlet graphs with 2*rt* = *rm* vertices.The join of these 2 sunlet graphs gives 4 regular graphs which can be decompose into 2 hamilton cycles which follows from the result of Bermond etal [9].

3.8 Theorem

Let *r, m, n* be positive even integers satisfying 3 < *r* · *m*. If $C_r[\overline{K}_m]$, can be decomposed into sunlet graph L_n of length *n* = *rm* for all *m* in the range $r \leq m$ < 2*r* with $m(m-2) \equiv 0(\bmod r)$,then $C_r[\overline{K}_m]$ can be decomposed into sunlet graph L_n of length *n* for all $m \geq r$ with $m(m-2) \equiv 0(\bmod r)$.

Proof:

Suppose that $C_r[\overline{K}_m]$ can be decomposed into sunlet graphs L_n whenever $m(m-2) \equiv 0(\bmod r)$ and $r \leq m$ < 2*r*.Let *m* and *r* be positive even integers such that $m(m-2) \equiv 0(\bmod r)$.Let *m* = *pr* + *t* for integers *p* with 0 < *t* < *r* .Observe that $m(m-2) \equiv 0(\bmod r)$ implies that *m* or $m-2$ is a multiple of *r*. If *m* = *pr*+*t* then *t* is either 0 or 2 .Since *r* is even, then $r(r-2) \equiv 0(\bmod t)$ as well .Partition the edge set of $C_r[\overline{K}_m]$ into *m* sets such that each set induces a subgraph isomorphic to sunlet graph L_n in such a way that $\frac{mr}{t}$ vertices will have degree 3 and $\frac{mr}{t}$ vertices have degree 1.If there exist a vertex x_{ij} such that the degree of the vertex x_{ij} is either 3 or 1.i.e for L_n to decompose

$$C_r[\overline{K}_m], \sum_{i=1}^{n} \deg(x_{ij}) = 2m, x_{ij} \in V(L_n)$$

which follows from lemma (3.2).We now use induction on *p*.It is true from lemma (3.2) that L_n decomposes $C_r[\overline{K}_m]$ for *p* = 1.Suppose *p* > 1,assume that it is true for *p* = *k*.Suppose *p* = *k*+1,then *m* = (*k* + 1) *r* + *t*.

$$,\sum_{i=1}^{n} \deg(x_{ij}) = \frac{3((k+1)r+t)}{2} + \frac{(k+1)r+t}{2}, = 2((k+1)r+t) = 2m$$

It is true for *p* = *k*+ 1,therefore it is true for all integer *p*. Therefore L_n decomposes $C_r[\overline{K}_m]$ for all *m* > *r*.Hence the result.

4 Main result

4.1 Theorem

Let $G = C_r[\overline{K}_m]$ with rm^2 edges *r*, *m* even, *n* = *rm* ,let *H* be a collection of *m* disjoint sunlet graphs on 2*n* vertices .Then $G(2) = L'_{2n} \oplus L''_{2n}$ where $L'_{2n} \approx L''_{2n} \approx H$.

Therefore $H \mid G(2)$.

Proof:

First show that $G(2) = L'_{2n} \oplus L''_{2n}$. From lemma (3.2), L_n decomposes $C_r[\overline{K}_m]$ into *m* sunlet graph with *n* vertices.i.e $\cup_{k=1}^{m} L_{ni}$ decomposes $C_r[\overline{K}_m] = G$ .Next show that L_{n2k} decomposes *G*(2).It is sufficient to show that $L_{nk}(2)$ can be decompose into 2 copies of L_{n2k} , *k* = 1,...,*m*. First form 2 cycles C'_{nk} and C''_{nk} with vertex set x'_{ij} and x''_{ij} .Define a mapping *f* by $f(x'_{ij}) = y'_{ij}$ and $f(x''_{ij}) = y''_{ij}$ where y'_{ij}, y''_{ij} represent the pendant vertices of $L_{nk}(2)$ .Attached each pendant vertices $f(x'_{ij}), f(x''_{ij})$ to each vertex x'_{ij}, x''_{ij} in C'_{nk}, C''_{nk} respectively. C'_{nk}, C''_{nk} with pendant vertices $f(x'_{ij}), f(x''_{ij})$ attached respectively gives 2 sunlet graphs L'_{2nk}, L''_{2nk} . Therefore $L_{nk}(2)$ can be decompose into 2 sunlet graph L'_{2nk}, L''_{2nk} .i.e $L_{nk}(2) = L'_{2nk} \oplus L''_{2nk}$ .Therefore

$$G(2) = \bigcup_{k=1}^{m} L_{nk}(2) = \bigcup_{k=1}^{m} L'_{2nk} \oplus \bigcup_{k=1}^{m} L''_{2nk} = L'_{2n} \oplus L''_{2n}.$$

Where $L_{2n} = \bigcup_{k=1}^{m} L_{2nk}$ .Hence $H \mid G(2)$ since $L'_{2n} \approx L''_{2n} \approx H$.

4.2. Theorem

Let $C_r[\overline{K}_m] = G$ with rm^2 edges, *r* ,*m* even .Let *H* be a collection of *m* disjoint sunlet graphs L_n, *n* = *rm.* Then

$G(2) = G^1 \oplus G^2 \oplus G^3 \oplus G^4$

where

$G^1 \approx G^2 \approx G^3 \approx G^4 \approx H$.

Therefore $H \mid G(2)$

.

Proof:

Assume that *H* consists of disjoint sunlet graphs L_n with lengths $n_1, n_2, ..., n_{nm}$

Where $m \sum_{k=1} rm^2 = nm$. Let *G* consist of set of vertices $x_{ij}, i = 1,...,r$

And $j = 0,1,...,m-1$

with rm^2 edges and *G*(2) be a graph with $4rm^2$ edges and 2*rm* vertices.Let G^1, G^2, G^3, G^4 be the subgraph of *G*(2) with equal number of vertices and edges i.e

G^1 contains the set of vertices $\{x'_{ij}\}$ with edge set $\{x'_{ij} x'_{i+1j}\}$.

G^2 contains the set of vertices $\{x''_{ij}\}$ with edge set $\{x''_{ij} x''_{i+1j}\}$.

G^3 contains the set of vertices $\{x'_{ij}, x''_{i+1j}\}$ with edge set $\{x'_{ij} x''_{i+1j}\}$ *2*

G^4 contains the set of vertices $\{x''_{ij}, x'_{i+1j}\}$ with edge set $\{x''_{ij} x'_{i+1j}\}$

Since each $G^a, a = 1,2,3,4$ consists of rm^2 edges then G^a partition *G*(2) that is

$G(2) = G^1 \oplus G^2 \oplus G^3 \oplus G^4$.

Each G^a has sunlet graph decomposition which follows from lemma (3.2).Then

$$G^a = \bigcup_{k=1}^{m} L^a{}_{nk}$$

Therefore $G^a \approx H$

Hence

$H \mid G(2)$.

4.3 Corollary

Let $C_r[\overline{K}_m] = G$ be a graph with sunlet graph decomposition (that is a decomposition into edge-disjoint sunlet graphs).Then any proper even list of sunlet graphs L_n, *n* = *rm* packs *G*(2).

Proof:

A sunlet graph in *G*(2) is a cycle together with pendant vertices attached to each vertex of the cycle.Assume that

$$\partial = (L_{n1}, L_{n1}, L_{n1}, L_{n1}, L_{n2}, L_{n2}, L_{n2}, L_{n2}, L_{nm}, L_{nm}, L_{nm}, L_{nm})$$

is a proper even list of sunlet graphs on *n* vertices.Where *n* is the order of *G*.

$L_{n1} \oplus L_{n2} \oplus \ldots \oplus L_{nm}$ be a sunlet graph decomposition of *G*.

Let $L_{nk} \approx L_n, k = 1,\ldots,m$ .By theorem (4.2),

$$L_{nk}(2) = \bigcup_{k=1}^{m} L^a{}_{nk}, a = 1,2,3,4 \text{ i.e}$$

$$L_{nk}(2) = L^1{}_{nk} \oplus L^2{}_{nk} \oplus L^3{}_{nk} \oplus L^4{}_{nk}$$

where $L^1{}_{nk} \approx L^2{}_{nk} \approx L^3{}_{nk} \approx L^4{}_{nk}$ Whence

$$G(2) = L_{n1}(2) \oplus L_{n2}(2) \oplus \ldots \oplus L_{nm}(2)$$
$$= L^1{}_{n1} \oplus L^2{}_{n1} \oplus L^3{}_{n1} \oplus L^4{}_{n1} \oplus L^1{}_{n2} \oplus L^2{}_{n2} \oplus L^3{}_{n2}$$
$$\oplus L^4{}_{n2} \oplus \ldots \oplus L^1{}_{nm} \oplus L^2{}_{nm} \oplus L^3{}_{nm} \oplus L^4{}_{nm}$$

Therefore proper even list of sunlet graphs packs *G*(2) .Hence the result .

4.4 Theorem

If the graph $C_r[\overline{K}_m]$ decomposes into sunlet graph L_n, *n* = *rm, r,m* even, then the graph $C_r[\overline{K}_{ml}]$ decomposes into sunlet graphs L_n for any positive integer l.

Proof:

From the previous observation L_n decompose $C_r[\overline{K}_m]$, *n* = *rm, r,m* even.

Next show that $L_n[\overline{K}_l]$, can be ndecompose into l^2, copies of L_n.

Label the vert ices of $L_n[\overline{K}_l]$, as $x^a{}_{ij}, 1 \le a \le l$,Then form the sunlet graphs $L^1{}_n,\ldots,L^{l^2}{}_n$, from $L_n[\overline{K}_l]$, as follows:

Form the cycle $C_{\frac{n}{2}}$ from $l \times l$, latin square as

$$(1,u)(2,v)(3,u)\ldots(\frac{m}{2}-1,v)(\frac{n}{2},\alpha),$$

where *u* is the row ,*v* is the column and α is the entry in the latin square. Next join each vertex of the cycle to pendant vertices $x_{ij}{}^l$ for $l = \alpha$ in order to get sunlet graph L_n.Therefore $L_n[\overline{K}_l] = L_n{}^1 \oplus \ldots \oplus L^{l^2}{}_n$ .Hence sunlet graph L_n decomposes $C_r[\overline{K}_{ml}]$ for any positive integer *l* .

4.5 Corollary

Let $G = C_r[\overline{K}_m]$ be a graph with sunlet graph decomposition then any even list of sunlet graph L_n packs $G(l)$ for any positive integer *l*.

Proof:

Assume that $C_r[\overline{K}_m]$ decomposes into *m* sunlet graph L_n, from theorem (4.4) $L_n(l)$ decomposes into l^2 sunlet graph L_n for any positive integer *l*.

$$C_r[\overline{K}_{ml}]$$
$$= L_{n1}(l) \oplus L_{n2}(l) \oplus \ldots \oplus L_{nm}(l)$$
$$= L_{n1}{}^1 \oplus \ldots \oplus L_{n1}{}^{l^2} \oplus \ldots \oplus L_{nm}{}^1 \oplus \ldots \oplus L^{l^2}{}_{nm}$$

Which is an even list of L_n.Hence any proper even list of sunlet graph L_n packs $G(l)$ for any positive integer *l* .

4.6 Theorem

If the graph $C_r[\overline{K}_m]$ decomposes into sunlet graph L_n, *n* = *rm* , *m* even, then the graph $C_r[\overline{K}_{ml}]$ decomposes into sunlet graph L_{nl} for any positive integer *l*.

Proof:

Suppose $C_r[\overline{K}_m]$ decomposes into *m* sunlet graphs L_n which follows from

ISSN 2229-5518

lemma (3.2)and (3.3).It is sufficient to show that $L_n(l)$ can be decompose into sunlet graph L_{nl} where L_{nl} is a sunlet graph on nl vertices.Next form sunlet graph L_{nl} from $L_n(l)$.The cycle in L_n give rise to $C_{\frac{n}{2}}[\overline{K}_l]$ in $L_n(l)$.

$C_{\frac{n}{2}}[\overline{K}_l]$ can be decompose into *l* cycles $C_{\frac{nl}{2}}$ by lemma (3.2),(3.3).Then join each vertex in each *l* cycles $C_{\frac{nl}{2}}$ to the pendant vertices $x^l{}_{ij}$ to get l sunlet graphs L_{nl} i.e

$$L_n(l) = L_{nl1} \oplus L_{nl2} \oplus L_{nll}$$

Hence L_{nl} decomposes $C_r[\overline{K}_{ml}]$ for any positive integer l.

4.7 Corollary

Let $G = C_r[\overline{K}_m]$ be a graph with sunlet graph decomposition,then any proper even list of sunlet graph L_{nl} packs $G(l)$ for any positive integer l.

Proof:

Assume that proper list of sunlet graphs $\delta = (L^1{}_{nl1}, L^2{}_{nl1},..., L^l{}_{nl1}, L^1{}_{nl2}, ..., L^l{}_{nl2},..., L^1{}_{nlm},..., L^l{}_{nlm}$

,

is the proper list of sunlet graphs L_{nl}.Let $L_{n1} \oplus L_{n2} \oplus ... \oplus L_{nm}$ be a sunlet graph decomposition of *G*.i.e $L_{nk} \approx L_n, k = 1,2,...,m$. By theorem (4.6),

$$L_{nk}(l) = L^1{}_{nlk} \oplus L^2{}_{nlk} \oplus ... \oplus L^l{}_{nlk}.$$

$$L^1{}_{nlk} \approx L^2{}_{nlk} \approx ... \approx L^l{}_{nlk}$$

Whence

$$G(l) = L_{n1}(l) \oplus L_{n2}(l) \oplus ... \oplus L_{nm}(l)$$

$$= (L^1{}_{nl1} \oplus L^2{}_{nl1} \oplus ... \oplus L^l{}_{nl1} \oplus L^1{}_{nl2} \oplus ... \oplus L^l{}_{nl2} \oplus ... \oplus L^1{}_{nlm} \oplus ... \oplus L^l{}_{nlm}$$

Therefore any even list of sunlet graphs L_{nl} packs $G(l)$.

4.8 Theorem

The graph $C_r[\overline{K}_m]$ can be decompose into sunlet graph L_n, if and only if *n* divides rm^2 , $n = rm$ and *m* is even.

Proof:

The necessary condition follows from lemma (3.6) that is the number of edges in $C_r[\overline{K}_m]$ is rm^2 ,so *n* must divides rm^2 for a decomposition to occur.If *n* > *rm*,a vertex must be repeated within each sunlet graph which is impossible. If *n* < *rm*,then there is a vertex which does not appear in the sunlet graph which is also impossible for such decomposition to occur.So *n* = *rm*.

To show that if $n \mid rm^2$ then L_n decomposes $C_r[\overline{K}_m]$, it is sufficient to show that if $r \mid n$ and $m \mid n$ then $n \mid rm^2$.

Case 1: *r* divides *n*

Write $n = rq^2t$ where *t, q* is a positive integer and from lemma (3.6), $rq^2t \mid rm^2$,so $qt \mid m$,and we can write $m = qq't$ .Also $rq^2t = rqq't$,so $q = q'$.Next consider $C_r[\overline{K}_t]$, the degree of each vertex is 2*t* which is even.Each vertex appears *t* times in the sunlet graph L_{rt} decomposition which follows from lemma (3.2) and lemma (3.3).By theorem (4.4),the graph $C_r[\overline{K}_{qq't}]$ can be decompose into sunlet graph L_{rt} .Applying theorem (4.6),we have that $C_r[\overline{K}_{qq't}]$ decomposes into sunlet graph $L_{rqq't} = L_{rq^2t}$ for $q = q'$.Hence if *r* divides *n* then $C_r[\overline{K}_m]$ has sunlet graph decomposition.

Case 2: *m* divides *n*

Write $n = mt'$ for any positive integer $t' \geq 3$ and $m = qq'$

Case 2a: Suppose $m \equiv 2 (\bmod 4)$ and let *q* = 2,then $q' \equiv 1 (\bmod q)$.By lemma (3.1) $C_{t'}[\overline{K}_{qq'}]$ decomposes into sunlet graph $L_{t'q}$ and by theorem (4.6), $C_{t'}[\overline{K}_{qq'}]$

decomposes into sunlet graph $L_{t'qq'} = L_{t'm}$ .Hence sunlet graph L_n decomposes $C_r[\overline{K}_m]$ for $r = t'$.

Case 2b: Suppose $m \equiv 0(\bmod 4)$ .Let q = 2,then $q' \equiv 0(\bmod q)$. $C_{t'}[\overline{K}_{qq'}]$ decomposes into sunlet graph $L_{t'q}$ by lemma (3.1),also $C_{t'}[\overline{K}_{qq'}]$ decomposes into sunlet graph $L_{t'qq'} = L_{t'm}$ by theorem (4.6).Hence sunlet graph L_n decomposes $C_r[\overline{K}_m]$ for $r = t'$. Since sunlet graph $L_{rqq'}$, decomposes $C_r[\overline{K}_{qq'}]$ for both q' = positive odd integer and q' =positive even integer,therefore it is true for all positive integer q' .Hence if m divides n then $C_r[\overline{K}_m]$ has sunlet graph decomposition. Since q = 2 all through for $m = qq'$,it follows that for sunlet graph L_n to decompose $C_r[\overline{K}_m]$,m must be an even integer which also follows from lemma (3.4) .

References

[1] B. Alspach, H. Gavlas, Cyle decomposition of *Kn* and *Kn ¡ I*,J. Combin. Theory Ser.B,81 (2001), 77-99.

[2] M. Sajna, Cycle decomposition of *Kn* and *Kn ¡ I* ,Ph.D Thesis,Simon Fraser University 1999.

[3] N.J. Cavenagh, E.J. Billington, Decomposition of complete multipartite graphs into cycles of even length;Graphs and Combinatorics (2000) 16, 49-65.

[4] C.C. Linder, C.A. Rodger, Decomposition into cycles II; Cycle systems in contemporary design theory: a collectuin of surveys, J.H. Dinitz and D.R. Stinson (Editors), Wiley,New York, 1992, 325-369.

[5] R. Haggkvist, A lemma on cycle decompositions; Annals of Discrete Mathematics 27 (1985), 227-232.

[6] D. Froncek, P. Kovar, M. Kubesa, Decomposition of complete graphs into blown-up cycles *Cm*[2]; Discrete Mathematics 310 (2010),1003-1015.

[7] R. Anitha, R.S. Lekshmi, N-sun decomposition of complete,complete bipartite and some harary graphs; Int. J. of Computational and Mathematical sciences 2 Winter (2008).

[8] D. Sotteau; Decomposition of *Km;n* ³*K*¤ *m;n* into cycles (circuit) of length 2*k*, Journal of Comb. Theory,Series B 30,(1981),75-81.

[9] J.C. Bermond, O. Favaron, M. Maheo, Hamilton decomposition of cayley graphs of degree 4;J. Comb. Theory Ser. B 46,(1989),142-153.

[10] N.J. Cavenagh, Decomposition of complete tripartite graphs into *k* cycles; Australasian J. of Combinatorics 18 (1998) 193-200.

[11] R. Laskar, Decomposition of some composite graphs into hamilton cycles, Pro. 5th Hungarian coll. keszthely 1976, North Holland, 1978, 705-716.

A. D. Akinola
Department of Mathematics,
College of Natural Science,
University of Agriculture,
Abeokuta,Ogun State, Nigeria.
email:abolaopeyemi@yahoo.co.uk